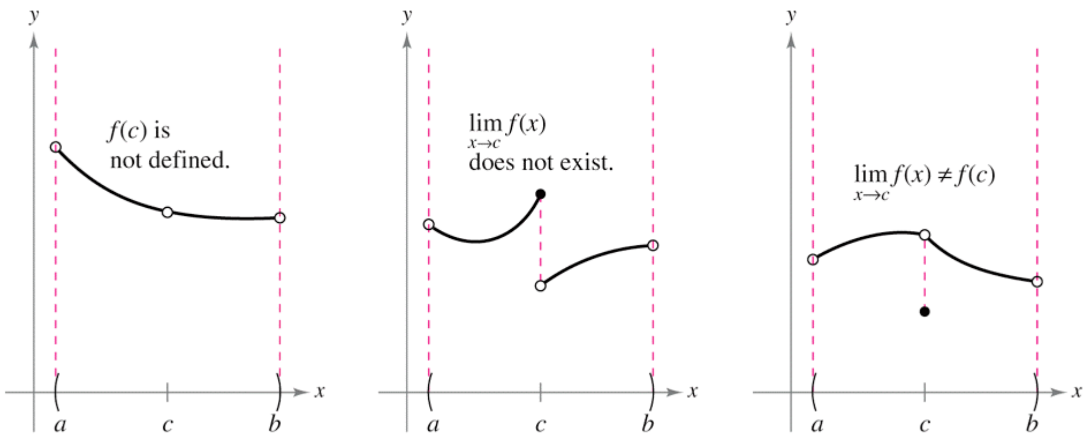
Math 150, *Lecture Notes- Bonds* Name____________

Section 1.4 Continuity and One-Sided Limits

Continuity at a Point and on an Open Interval



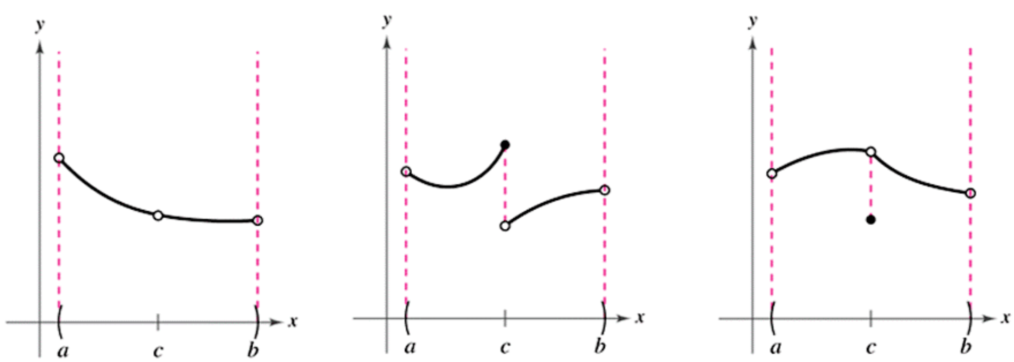
Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

Definition of Continuity

Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

Continuity on an Open Interval: A function is **continuous on an open interval (a, b)** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**



(a) Removable discontinuity **(b)** Nonremovable discontinuity **(c)** Removable discontinuity

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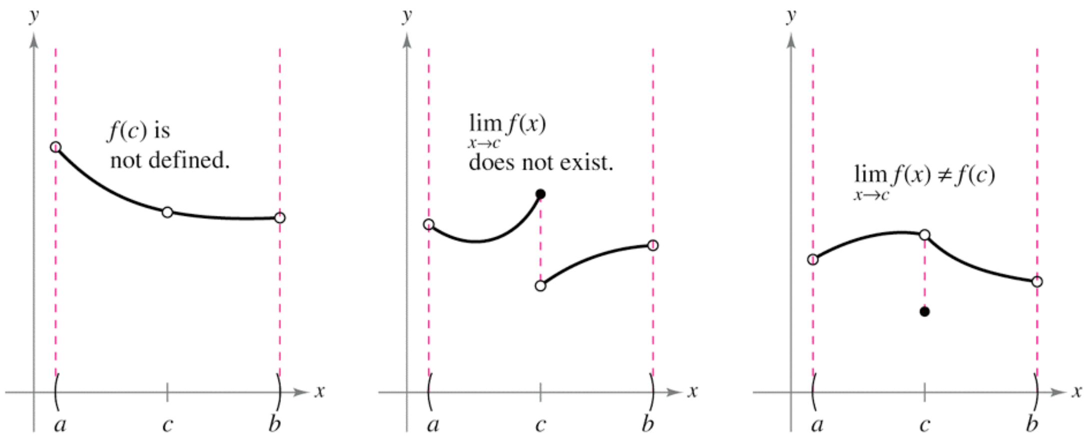
***Section 1.4** Continuity and One-Sided Limits*

Continuity at a Point and on an Open Interval



Three conditions exist for which the graph of f is not continuous at $x = c$.
Figure 1.25

Definition of Continuity This definition is on the First Test!

Continuity at a Point: A function f is **continuous at *c*** if the following three conditions are met.

1. $f(c)$ is defined.
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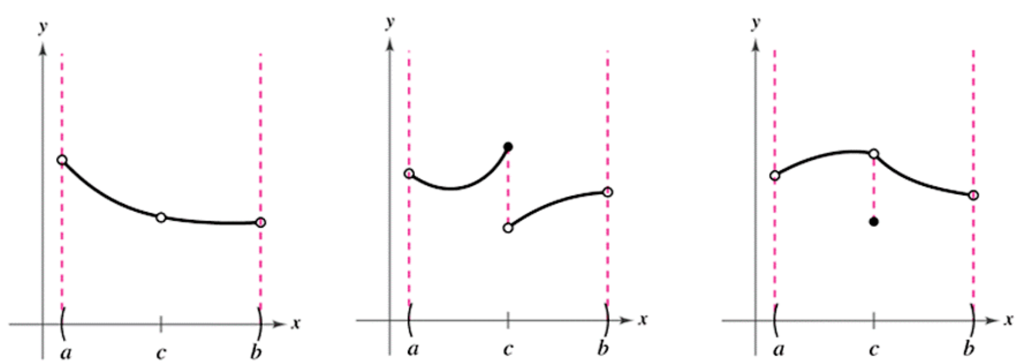


(a) Removable discontinuity **(b)** Nonremovable discontinuity **(c)** Removable discontinuity

Discuss the continuity of each function.

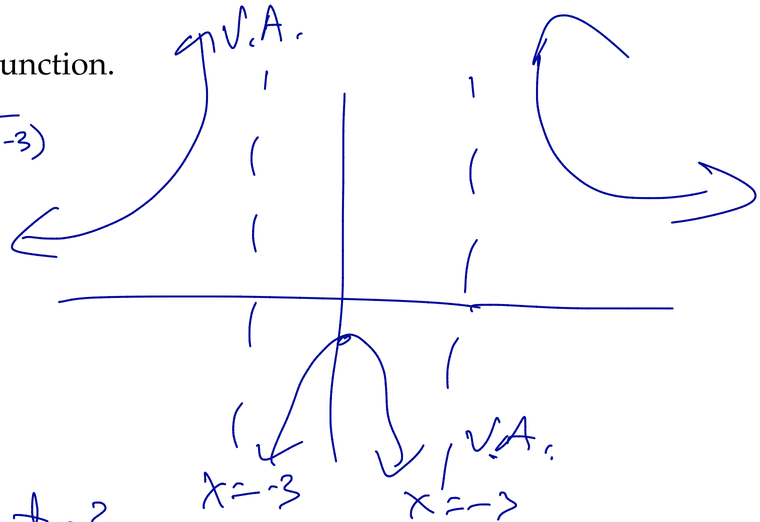
Ex.1 $f(x)=\dfrac{1}{x^2-9} = \dfrac{1}{(x+3)(x-3)}$

$(x+3)(x-3)=0$

either

$x+3=0$, or $x-3=0$

$\underline{x=-3}$ $\quad$ $\underline{x=3}$

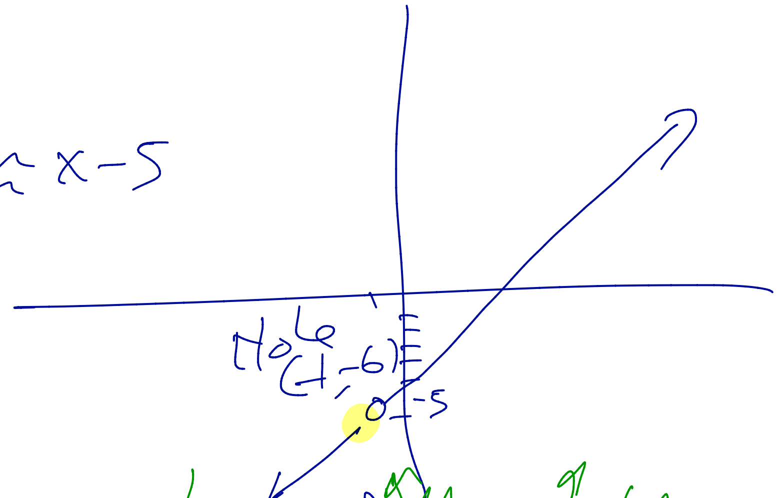


2 Non-Removable Discontinuities

one at 3 & one at -3

Ex.2 $g(x)=\dfrac{x^2-4x-5}{x+1}$

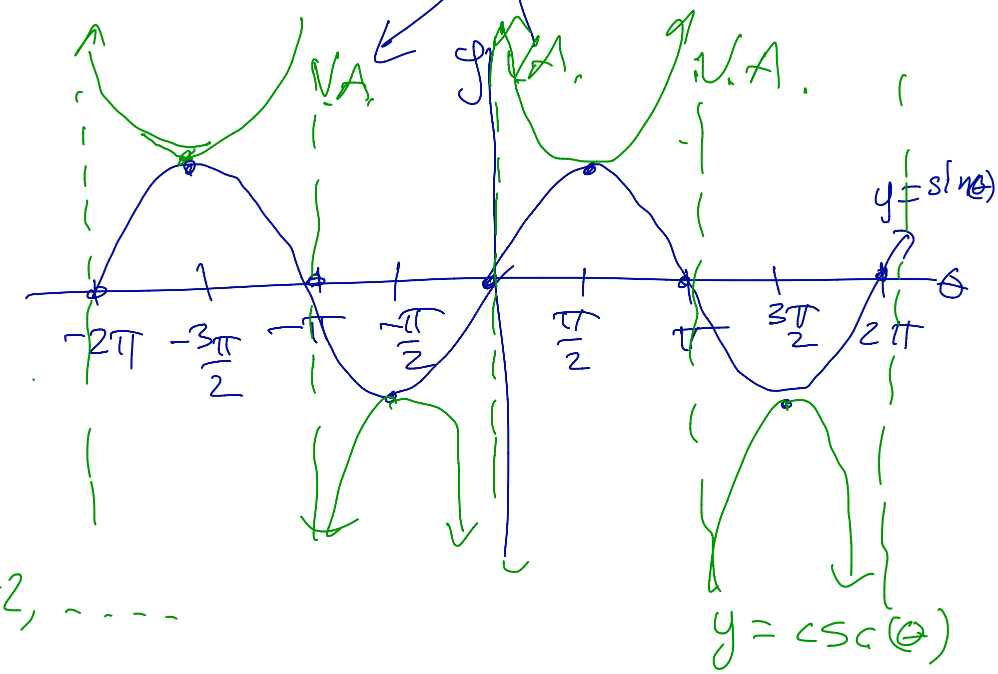
$g(x)=\dfrac{(x+1)(x-5)}{(x+1)} \approx x-5$



Removable Discontinuity at -1

Ex.3 $h(\theta)=\csc(\theta)$

$h(\theta)=\dfrac{1}{\sin(\theta)}$

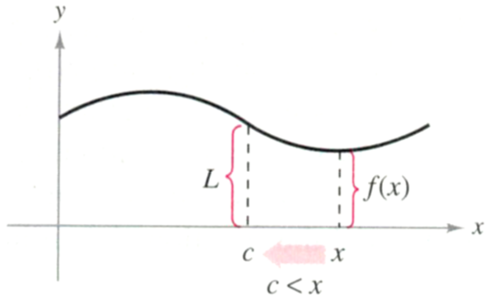


Non-removable Discontinuities

at $n\pi$,

where $n=0,\pm1,\pm2,\ldots$

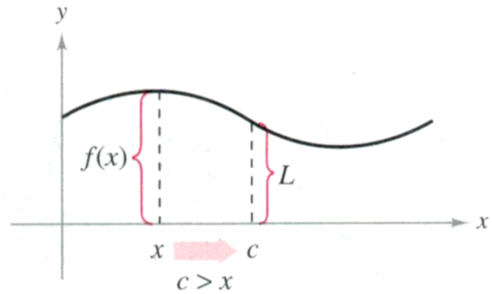
One-Sided Limits and Continuity on a Closed Interval



(a) Limit as x approaches c from the right.

$$\lim_{x \to c^+} f(x) = L.$$

limit from the right



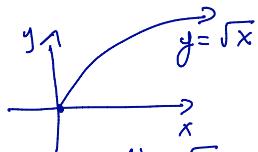
(b) Limit as x approaches c from the left.

$$\lim_{x \to c^-} f(x) = L.$$

limit from the left

Ex.4

$$\lim_{x \to 0^+} \sqrt[n]{x} = 0.$$

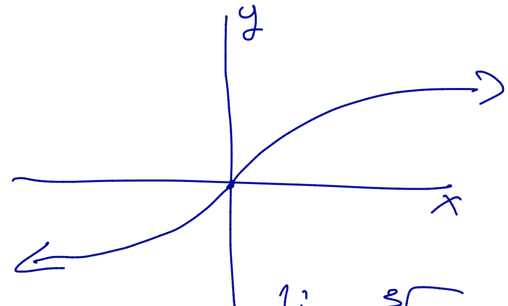


$\lim_{x \to 0} \sqrt{x} = $ D.N.E.

$\lim_{x \to 0^+} \sqrt{x} = 0$

$\lim_{x \to 0^-} \sqrt{x} = $ D.N.E.

$y = \sqrt[3]{x}$



$\lim_{x \to 0} \sqrt[3]{x} = 0$

$\lim_{x \to 0^+} \sqrt[3]{x} = 0$

$\lim_{x \to 0^-} \sqrt[3]{x} = 0$

Ex.5

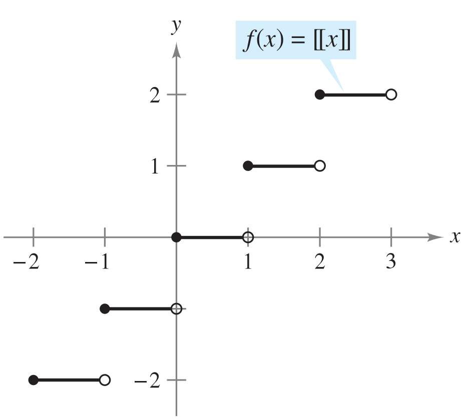
$$[\![x]\!] = \text{greatest integer } n \text{ such that } n \le x.$$

$\lim_{x \to -1} [\![x]\!] = $ D.N.E.

$\lim_{x \to -1^+} [\![x]\!] = -1$

$\lim_{x \to -1^-} [\![x]\!] = -2$

$\lim_{x \to -1^-} [\![x]\!] \ne \lim_{x \to -1^+} [\![x]\!]$



Greatest integer function

When the limit from the left is not equal to the limit from the right, the (two-sided) ***limit does not exist***.

THEOREM 1.10 The Existence of a Limit

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

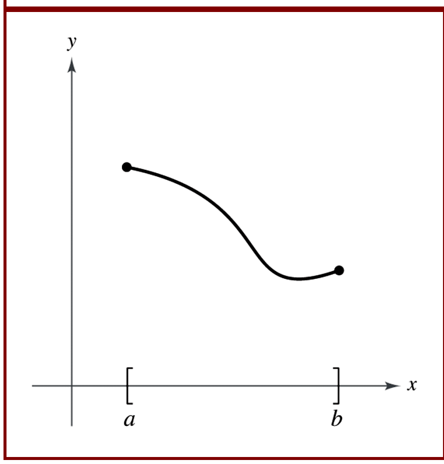
$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Definition of Continuity on a Closed Interval

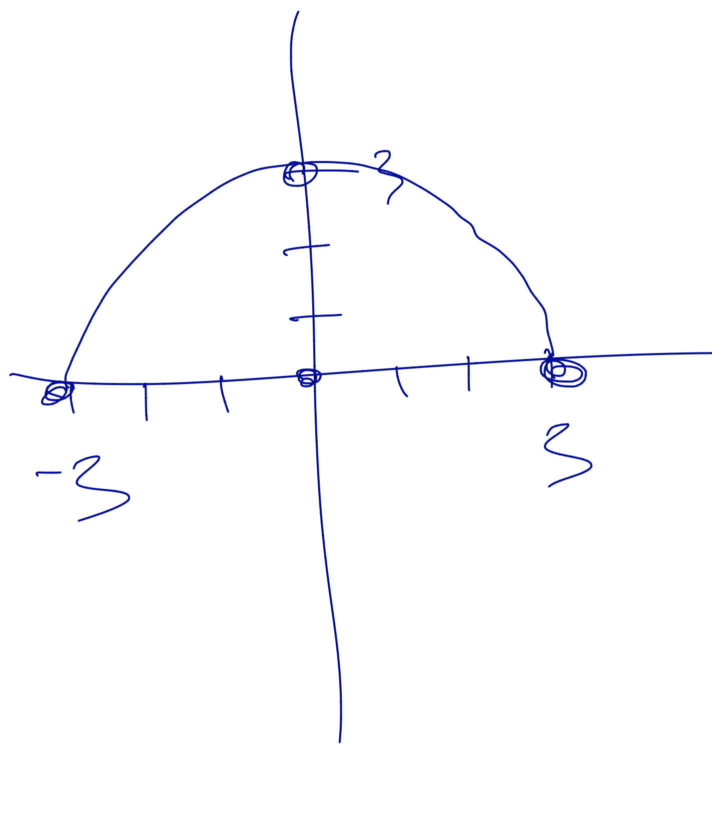
A function f is **continuous on the closed interval** $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).



Ex.6 $f(x) = \sqrt{9 - x^2}$



$$y = \sqrt{9 - x^2}$$

$$y^2 = 9 - x^2$$

$$x^2 + y^2 = 9$$

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$
2. Rational: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

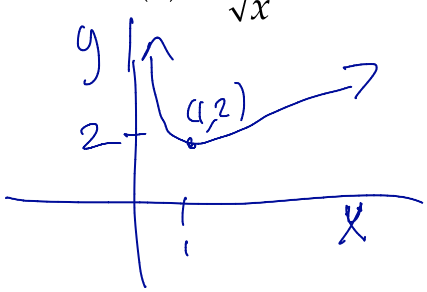
By combining Theorem 1.11 with this summary, you can conclude that a wide variety of elementary functions are continuous at every point in their domains.

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Ex.7 Describe the intervals on which the following functions are continuous.

(a) $f(x) = \dfrac{x+1}{\sqrt{x}}$



f is continuous on $(0, \infty)$

$\sqrt{x} = 0$

$(\sqrt{x})^2 = 0^2$

$x = 0$

$f(x) = \dfrac{x+1}{\sqrt{x}}$

Ratio & Composition

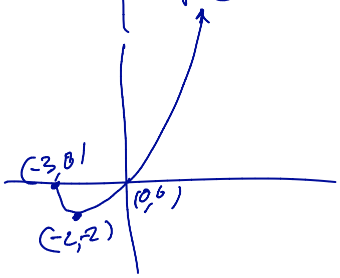
Product & composition

(b) $g(x) = x\sqrt{x+3}$

$x + 3 \geq 0$

$x \geq -3$

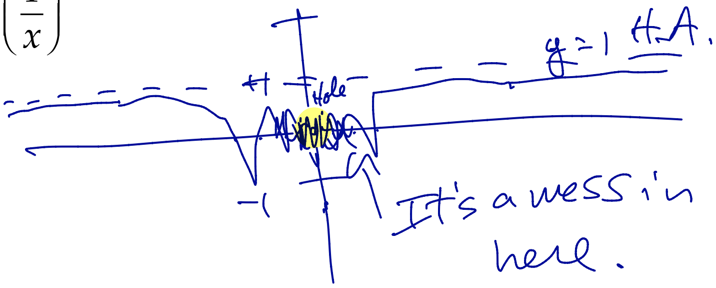
"cts" on $[-3, \infty)$



TRIG & composition

(c) $h(x) = \cos\left(\dfrac{1}{x}\right)$

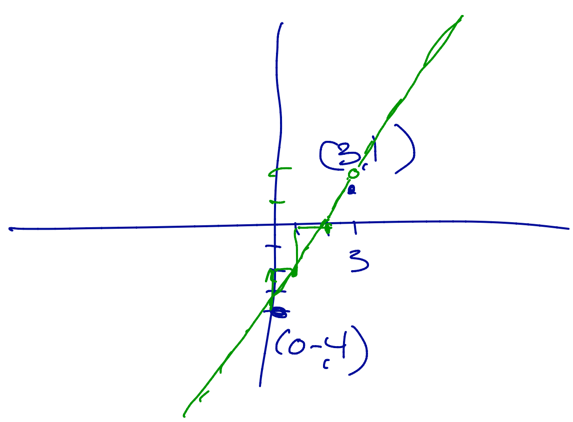
$x \neq 0$



(d) $f(x) = \begin{cases} 2x - 4, & x \neq 3 \\ 1, & x = 3 \end{cases}$

$y = mx + b$

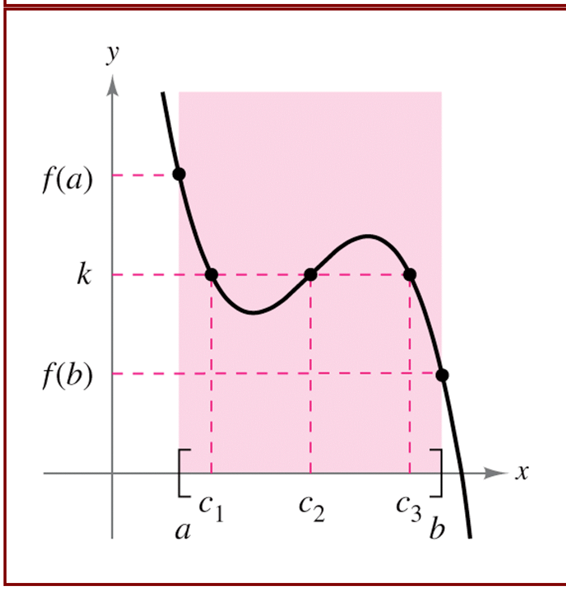
$m = \frac{2}{1}$, $b = -4$

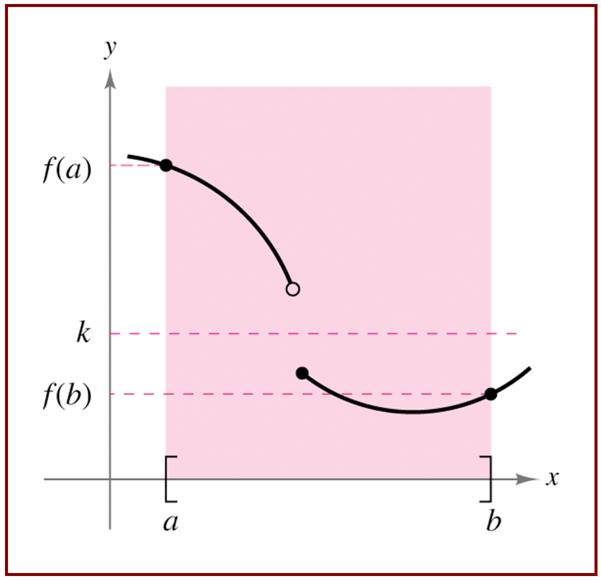


THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$





Ex.8 Verify that the Intermediate Value Theorem applies to $f(x) = x^2 - 6x + 8$ on $[0,3]$, and then the value of c guaranteed by the theorem, where $f(c) = 0$.

$k = f(c)$

$k = 0$

1st: $f(x) = x^2 - 6x + 8$ is a "cts" polynomial

2nd: $[0,3]$ is a closed interval,

3rd: check $f(a)$ & $f(b)$. Is k between?

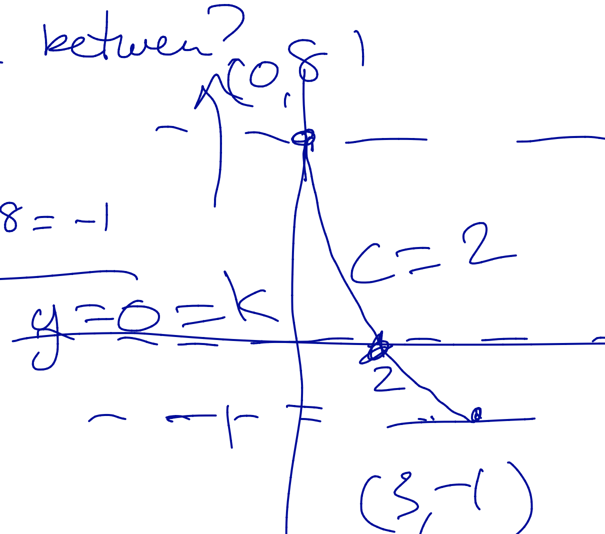
$f(a) = f(0) = (0)^2 - 6(0) + 8 = 8$

$f(b) = f(3) = (3)^2 - 6(3) + 8 = 9 - 18 + 8 = -1$

$-1 \le k \le 8$

$-1 \le 0 \le 8$

$f(3) \le k \le f(0)$



4th Then, The I.V.T. tells us that $\exists\, c \in [0,3]$ "there exists" such that $f(c) = k$, or $f(c) = 0$

$\longrightarrow c = 2.$

The output expectation and the output must match.

Ex.9 Find the constant a such that $f(x)=\begin{cases} 3x^3, & x\le 1 \;\text{left} \\ ax+5, & x>1 \;\text{Right} \end{cases}$ is continuous on the entire real number line.

$y=f(x)$ needs to "connect" at $x=1$.

$$\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$$

polynomial is "cts"

$$\lim_{x\to 1^-}\left[3x^3\right] = 3(1)^3 = \lim_{x\to 1^+}(ax+5)$$

"plug-in x-value"

$$3(1)^3 = 3 = a(1)+5$$

$$3 = a+5$$

$$-5+3 = a$$

$$-2 = a$$

$$f(x)=\begin{cases} 3x^3, & x\le 1 \\ -2x+5, & x>1 \end{cases}$$